# Miscellaneous Geometry Facts

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## Cyclic Quadrilaterals

1. A convex quadrilateral $ABCD$ is cyclic if and only if either:
   (a) $\angle ADB = \angle ACB$
   (b) $\angle DAB + \angle BCD = 180°$
2. The above two conditions can be restated as a single condition in terms of directed angles: Four points $A, B, C$ and $D$ are concyclic if and only if $\measuredangle ABC = \measuredangle ADC$.
3. (Power of a Point) Let $ABCD$ be a convex quadrilateral such that $AB$ and $CD$ intersect at $P$ and diagonals $AC$ and $BD$ intersect at $Q$. $ABCD$ is cyclic if and only if either:
   (a) $AQ \cdot QC = BQ \cdot QD$ or equivalently $QAD$ and $QBC$ are similar
   (b) $PA \cdot PB = PC \cdot PD$ or equivalently $PAD$ and $PCB$ are similar
4. Given a triangle $ABC$, the intersections of the internal and external bisectors of angle $\angle BAC$ with the perpendicular bisector of $BC$ both lie on the circumcircle of $ABC$.
5. (Ptolemy's Theorem) A quadrilateral $ABCD$ is cyclic if and only if

$$AB \cdot CD + AD \cdot BC = AC \cdot BD$$

6. Let $ABCD$ be a cyclic quadrilateral such that $AB$ and $CD$ intersect at $P$ and diagonals $AC$ and $BD$ intersect at $Q$. Then:

$$\frac{BQ}{QD} = \frac{AB \cdot BC}{AD \cdot DC} \quad \text{and} \quad \frac{PB}{PA} = \frac{BC \cdot BD}{AC \cdot AD}$$

7. (Polars) Let $ABCD$ be a cyclic quadrilateral inscribed in circle $\Gamma$ such that $AB$ and $CD$ intersect at $P$ and diagonals $AC$ and $BD$ intersect at $Q$. If the tangents drawn from $P$ to $\Gamma$ touch $\Gamma$ at $R$ and $S$, then $R$, $Q$ and $S$ are collinear.

## Circles

1. (Power of a Point) Given a circle $\Gamma$ with center $O$ and a point $P$ then for any line $\ell$ through $P$ that intersects $\Gamma$ at $A$ and $B$, the value $PA \cdot PB$ is constant as $\ell$ varies and is equal to the power of the point $P$ with respect to $\Gamma$.

   (a) The power of $P$ is equal to $r^2 - PO^2$ if $P$ is inside $\Gamma$ and $PO^2 - r^2$ otherwise.
   (b) If $PA$ is tangent to $\Gamma$, then the power of $P$ is equal to $PA^2$.

2. (Radical Axis) Given two circles $\Gamma_1$ and $\Gamma_2$, the set of all points $P$ with equal powers with respect to $\Gamma_1$ and $\Gamma_2$ is a line which is the radical axis of the two circles.
   (a) The radical axis is perpendicular to the line through the centers of $\Gamma_1$ and $\Gamma_2$.
   (b) If $\Gamma_1$ and $\Gamma_2$ intersect at $A$ and $B$, then the radical axis passes through $A$ and $B$.
   (c) If $AB$ is a common tangent with $A$ on $\Gamma_1$ and $B$ on $\Gamma_2$, then the radical axis passes through the midpoint of $AB$.

3. (Radical Center) Given three circles $\Gamma_1, \Gamma_2$ and $\Gamma_3$, the three radical axes between pairs of the three circles meet at a common point $P$ which is the radical center of the circles.

4. A point $P$ is a circle of radius zero and the radical axis of $P$ and a circle $\Gamma$ is the line through the midpoints of $PA$ and $PB$ where $A$ and $B$ are points on $\Gamma$ such that $PA$ and $PB$ are tangent to $\Gamma$.

5. (Monge's Theorem) Given three circles $\Gamma_1, \Gamma_2$ and $\Gamma_3$. If $P$, $Q$ and $R$ are the external centers of homothety between pairs of the three circles, then $P$, $Q$ and $R$ are collinear. If $P$ and $Q$ are internal centers of homothety, then $P$, $Q$ and $R$ are also collinear.

6. Two circles $\Gamma_1$ and $\Gamma_2$ intersect at $R$ and have centers $O_1$ and $O_2$. If $P$ and $Q$ are the internal and external centers of homothety between the two circles, then $\angle PRQ = 90^\circ$. The lines $RP$ and $RQ$ are the internal and external bisectors of $\angle O_1RO_2$.

## Triangle Geometry

1. (Angle Bisector Theorem) Let $ABC$ be a given triangle and let $P$ and $Q$ be the intersections of the internal and external bisectors of angle $\angle ABC$ with line $AC$. Then
$$\frac{AB}{BC} = \frac{AP}{PC} = \frac{AQ}{QC}$$

2. Angles around the centers of a triangle $ABC$:
   (a) If $I$ is the incenter of $ABC$ then $\angle BIC = 90^\circ + \frac{a}{2}$, $\angle IBC = \frac{b}{2}$ and $\angle ICB = \frac{c}{2}$.
   (b) If $H$ is the orthocenter of $ABC$ then $\angle BHC = 180^\circ - a$, $\angle HBC = 90^\circ - c$ and $\angle HCB = 90^\circ - b$.
   (c) If $O$ is the circumcenter of $ABC$ then $\angle BOC = 2a$ and $\angle OBC = \angle OCB = 90^\circ - a$.
   (d) If $I_a$ is the $A$-excenter of $ABC$ then $\angle AI_aB = \frac{c}{2}$, $\angle AI_aC = \frac{b}{2}$ and $\angle BI_aC = 90^\circ - \frac{a}{2}$.

3. Pedal triangles of the centers of a triangle $ABC$:
   (a) If $DEF$ is the triangle formed by projecting the incenter $I$ onto sides $BC$, $AC$ and $AB$, then $I$ is the circumcenter of $DEF$ and $\angle EDF = 90^\circ - \frac{a}{2}$.

(b) If $DEF$ is the triangle formed by projecting the orthocenter $H$ onto sides $BC$, $AC$ and $AB$, then $H$ is the incenter of $DEF$ and $\angle EDF = 180^\circ - 2a$.

(c) The medial triangle of $ABC$ is the pedal triangle of the circumcenter $O$ of $ABC$ and $O$ is its orthocenter.

4. Alternate methods of defining the orthocenter and circumcenter:

(a) $O$ is the circumcenter of $ABC$ if and only if $\measuredangle AOB = 2\measuredangle ACB$ and $OA = OB$.

(b) $H$ is the orthocenter of $ABC$ if and only if $H$ lies on the altitude from $A$ and satisfies that $\measuredangle BHC = 180^\circ - \measuredangle BAC$.

5. Facts related to the orthocenter $H$ of a triangle $ABC$ with circumcircle $\Gamma$:

(a) If $O$ is the circumcenter of $ABC$, then $\angle BAH = \angle CAO$.

(b) If $D$ is the point diametrically opposite to $A$ on $\Gamma$ and $M$ is the midpoint of $BC$, then $M$ is also the midpoint of $HD$.

(c) If $AH, BH$ and $CH$ intersect $\Gamma$ again at $D, E$ and $F$, then there is a homothety centered at $H$ sending the pedal triangle of $H$ to $DEF$ with ratio 2.

(d) If $D$ and $E$ are the intersections of $AH$ with $BC$ and $\Gamma$, respectively, then $D$ is the midpoint of $HE$.

(e) $H$ lies on the three circles formed by reflecting $\Gamma$ about $AB$, $BC$ and $AC$.

(f) If $M$ is the midpoint of $BC$ then $AH = 2 \cdot OM$.

(g) If $BH$ and $CH$ intersect $AC$ and $AB$ at $D$ and $E$, and $M$ is the midpoint of $BC$, then $M$ is the center of the circle through $B, D, E$ and $C$, and $MD$ and $ME$ are tangent to the circumcircle of $ADE$.

6. Facts related to the incenter $I$ and excenters $I_a, I_b, I_c$ of $ABC$ with circumcircle $\Gamma$:

(a) If the incircle of $ABC$ is tangent to $AB$ and $AC$ at points $D$ and $E$ and $s$ is the semiperimeter of $ABC$ then

$$AD = AE = \frac{AB + AC - BC}{2} = s - BC$$

(b) If $AI$ intersects $\Gamma$ at $D$ then $DB = DI = DC$, $D$ is the midpoint of $II_a$, and $II_a$ is a diameter of the circle with center $D$ which passes through $B$ and $C$.

(c) If $AI, BI$ and $CI$ intersect $\Gamma$ at $D, E$ and $F$, then $I_aI_bI_c$, $DEF$ and the pedal triangle of $I$ are similar and have parallel sides.

(d) $I$ is the orthocenter of $I_aI_bI_c$ and $\Gamma$ is the nine-point circle of $I_aI_bI_c$.

(e) If $BI$ and $CI$ intersect $\Gamma$ again at $D$ and $E$, then $I$ is the reflection of $A$ in line $DE$ and if $M$ is the intersection of the external bisector of $\angle BAC$ with $\Gamma$, then $DMEI$ is a parallelogram.

(f) If the incircle and $A$-excircle of $ABC$ are tangent to $BC$ at $D$ and $E$, $BD = CE$.

(g) If the $A$-excircle of $ABC$ is tangent to $AB$, $AC$ and $BC$ at $D$, $E$ and $F$ then $AB + BF = AC + CF = AD = AE = s$ where $s$ is the semi-perimeter of $ABC$.

   (h) If $M$ is the midpoint of arc $BAC$ of $\Gamma$, then $M$ is the midpoint of $I_bI_c$ and the center of the circle through $I_b, I_c, B$ and $C$.

7. (Nine-Point Circle) Given a triangle $ABC$, let $\Gamma$ denote the circle passing through the midpoints of the sides of $ABC$. If $H$ is the orthocenter of $ABC$, then $\Gamma$ passes through the midpoints of $AH, BH$ and $CH$ and the projections of $H$ onto the sides of $ABC$.

8. (Feuerbach's Theorem) The nine-point circle is tangent to the incircle and excircles.

9. (Euler Line) If $O$, $H$ and $G$ are the circumcenter, orthocenter and centroid of a triangle $ABC$, then $G$ lies on segment $OH$ with $HG = 2 \cdot OG$.

10. (Symmedian) Given a triangle $ABC$ such that $M$ is the midpoint of $BC$, the symmedian from $A$ is the line that is the reflection of $AM$ in the bisector of angle $\angle BAC$.

    (a) If the tangents to the circumcircle $\Gamma$ of $ABC$ at $B$ and $C$ intersect at $N$, then $N$ lies on the symmedian from $A$ and $\angle BAM = \angle CAN$.

    (b) If the symmedian from $A$ intersects $\Gamma$ at $D$, then $AB/BD = AC/CD$.

11. If the median from $A$ in a triangle $ABC$ intersects the circumcircle $\Gamma$ of $ABC$ at $D$, then $AB \cdot BD = AC \cdot CD$.

12. (Euler's Formula) Let $O$, $I$ and $I_a$ be the circumcenter, incenter and $A$-excenter of a triangle $ABC$ with circumradius $R$, inradius $r$ and $A$-exradius $r_a$. Then:

    (a) $OI = \sqrt{R(R-2r)}$.

    (b) $OI_a = \sqrt{R(R+2r_a)}$.

13. (Poncelet's Porism) Let $\Gamma$ and $\omega$ be two circles with centers $O$ and $I$ and radii $R$ and $r$, respectively, such that $OI = \sqrt{R(R-2r)}$. Let $A, B$ and $C$ be any three points on $\Gamma$ such that lines $AB$ and $AC$ are tangent to $\omega$. Then line $BC$ is also tangent to $\omega$.

14. (Apollonius Circle) Let $ABC$ be a given triangle and let $P$ be a point such that $AB/BC = AP/PC$. If the internal and external bisectors of angle $\angle ABC$ meet line $AC$ at $Q$ and $R$, then $P$ lies on the circle with diameter $QR$.

15. Let $ABC$ be a given triangle with incircle $\omega$ and $A$-excircle $\omega_a$. If $\omega$ and $\omega_a$ are tangent to $BC$ at $M$ and $N$, then $AN$ passes through the point diametrically opposite to $M$ on $\omega$ and $AM$ passes through the point diametrically opposite to $N$ on $\omega_a$.

16. Let $ABC$ be a triangle with incircle $\omega$ which is tangent to $BC$, $AC$ and $AB$ at $D$, $E$ and $F$. Let $M$ be the midpoint of $BC$. The perpendicular to $BC$ at $D$, the median $AM$ and the line $EF$ are concurrent.

17. Let $ABC$ be a triangle with incenter $I$ and incircle $\omega$ which is tangent to $BC$, $AC$ and $AB$ at $D$, $E$ and $F$. The angle bisector $CI$ intersects $FE$ at a point $T$ on the line adjoining the midpoints of $AB$ and $BC$. It also holds that $BFTID$ is cyclic and $\angle BTC = 90°$.

18. Let $ABC$ be a triangle with incircle $\omega$ and let $D$ and $E$ be the points at which $\omega$ is tangent to $BC$ and the $A$-excircle is tangent to $BC$. Then $AE$ passes through the point diametrically opposite to $D$ on $\omega$.

19. Let $ABC$ be a triangle with $A$-excenter $I_A$ and altitutde $AD$. Let $M$ be the midpoint of $AD$ and let $K$ be the point of tangency between the incircle of $ABC$ and $BC$. Then $I_A, K$ and $M$ are collinear.

## Collinearity and Concurrency

1. (Ceva's Theorem) Let $ABC$ be a triangle and $D$, $E$ and $F$ be on the lines $BC$, $AC$ and $AB$ such that an even number are on the extensions of the sides (zero or two). Then $AD$, $BE$ and $CF$ are concurrent if and only if

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

2. (Menelaus' Theorem) Let $ABC$ be a triangle and $D$, $E$ and $F$ be on the lines $BC$, $AC$ and $AB$ such that an odd number are on the extensions of the sides (one or three). Then $D$, $E$ and $F$ are collinear if and only if

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

3. (Trig Ceva) Let $ABC$ be a triangle and $D$, $E$ and $F$ be on the lines $BC$, $AC$ and $AB$ such that an even number are on the extensions of the sides (zero or two). Then $AD$, $BE$ and $CF$ are concurrent if and only if

$$\frac{\sin(\angle ABE)}{\sin(\angle CBE)} \cdot \frac{\sin(\angle BCF)}{\sin(\angle ACF)} \cdot \frac{\sin(\angle CAD)}{\sin(\angle BAD)} = 1$$

4. (Casey's Theorem) If $A_1, B_1$ and $C_1$ are points on the sides $BC, AC$ and $AB$ of a triangle $ABC$, then the perpendiculars to their respective sides at these three points are concurrent if and only if $BA_1^2 - CA_1^2 + CB_1^2 - AB_1^2 + AC_1^2 - BC_1^2 = 0$.

5. (Pascal's Theorem) If $A, B, C, D, E, F$ are points on a circle then the intersections of the pairs of lines $AB$ and $DE$, $BC$ and $EF$, $CD$ and $FA$ lie on a line.

6. (Pappus' Theorem) If $A, C$ and $E$ lie on one line $\ell_1$ and $B, D$ and $F$ lie on a line $\ell_2$, then the intersections of the pairs of lines $AB$ and $DE$, $BC$ and $EF$, $CD$ and $FA$ lie on a line.

7. (Brianchon's Theorem) If $ABCDEF$ is a hexagon with an inscribed circle then $AD$, $BE$ and $CF$ are concurrent.

8. (Desargues Theorem) Let $ABC$ and $XYZ$ be triangles. Let $D, E, F$ be the intersections of the pairs of lines $AB$ and $XY$, $BC$ and $YZ$, $AC$ and $XZ$. Then $D$, $E$ and $F$ are collinear if and only if $AX$, $BY$ and $CZ$ are concurrent.

9. Pascal's theorem is true when points are not necessarily distinct and many of its applications concern tangent lines when some of the six points are equal.

## Trigonometry

1. (Sine Law) Given a triangle $ABC$ with circumradius $R$

$$\frac{BC}{\sin \angle A} = \frac{AC}{\sin \angle B} = \frac{AB}{\sin \angle C} = 2R$$

2. (Cosine Law) Given a triangle $ABC$

$$BC^2 = AB^2 + AC^2 - 2 \cdot AB \cdot AC \cdot \cos \angle A$$

3. (Pythagorean Theorem) If $ABC$ is a triangle, then $\angle ABC = 90°$ if and only if

$$AB^2 + BC^2 = AC^2$$

4. Given a triangle $ABC$ and a point $D$ on line $BC$, then

$$\frac{\sin \angle BAD}{\sin \angle CAD} = \frac{BD \cdot AC}{CD \cdot AB}$$

5. (Stewart's Theorem) Let $a, b, c$ be the side lengths of a triangle $ABC$ and let $d$ be the length of a cevian from $A$ to $BC$ that divides $BC$ into segments of lengths $m$ and $n$ with $m$ closer to $B$. Then

$$b^2m + c^2n = a(d^2 + mn)$$

## Miscellaneous Synthetic Facts

1. (Spiral Similarity) Let $OAB$ and $OCD$ be directly similar triangles. Then $OAC$ and $OBD$ are also directly similar triangles.

2. The unique center of spiral similarity sending $AB$ to $CD$ is the second intersection of the circumcircles of $QAB$ and $QCD$ where $AC$ and $BD$ intersect at $Q$.

3. Lines $AB$ and $CD$ are perpendicular if and only if $AC^2 - AD^2 = BC^2 - BD^2$.

4. (Apollonius Circle) Given two points $A$ and $B$ and a fixed $r > 0$, then the locus of points $Q$ such that $AQ/BQ = r$ is a circle $\Gamma$ with center at the midpoint of $Q_1Q_2$ where $Q_1$ and $Q_2$ are the two points on line $AB$ satisfying $AQ_i/BQ_i = r$ for $i = 1, 2$.

5. Let $ABCD$ be a convex quadrilateral. The four interior angle bisectors of $ABCD$ are concurrent and there exists a circle $\Gamma$ tangent to the four sides of $ABCD$ if and only if $AB + CD = AD + BC$.

6. (Simson Line) Let $M$, $N$ and $P$ be the projections of a point $Q$ onto the sides of a triangle $ABC$. Then $Q$ lies on the circumcircle of $ABC$ if and only if $M$, $N$ and $P$ are collinear. If $Q$ lies on the circumcircle of $ABC$, then the reflections of $Q$ in the sides of $ABC$ are collinear and pass through the orthocenter of the triangle.

7. (Broken Chord Theorem) Let $E$ is the midpoint of major arc $\widehat{ABC}$ of the circumcircle of a triangle $ABC$ where $AB < BC$. If $D$ is the projection of $E$ onto $BC$, then $AB + BD = DC$.

8. (Butterfly Theorem) Let $M$ be the midpoint of a chord $XY$ of a circle $\Gamma$. The chords $AB$ and $CD$ pass through $M$. If $AD$ and $BC$ intersect chord $XY$ at $P$ and $Q$, then $M$ is also the midpoint of $PQ$.

9. (Miquel Point) Let $D$, $E$ and $F$ be points on sides $BC$, $AC$ and $AB$ of a triangle $ABC$. Then the circumcircles of $AEF$, $BDF$ and $CDE$ are concurrent.

10. (Isogonal Conjugates) Let $ABC$ be a triangle and $P$ be a point. If the reflection of $BP$ in the angle bisector of $\angle ABC$ and the reflection of $CP$ in the angle bisector $\angle ACB$ intersect at $Q$, then $Q$ lies on the reflection of $CP$ in the angle bisector of $\angle ACB$.

11. (Casey's Theorem) Let $O_1, O_2, O_3, O_4$ be four circles tangent to a circle $O$. Let $t_{ij}$ be the length of the external common tangent between $O_iO_j$ if $O_i$ and $O_j$ are tangent to $O$ from the same side and the length of the internal common tangent otherwise. Then

$$t_{12} \cdot t_{34} + t_{41} \cdot t_{23} = t_{13} \cdot t_{24}$$

The converse is also true: if the above equality holds then $O_1, O_2, O_3, O_4$ are tangent to $O$.

12. (Transversal Theorem) If $A$, $B$ and $C$ are collinear and $A', B'$ and $C'$ are points on $AP, BP$ and $CP$, then $A', B'$ and $C'$ are collinear if and only if

$$BC \cdot \frac{AP}{A'P} + CA \cdot \frac{BP}{B'P} + AB \cdot \frac{CP}{C'P} = 0$$

where all lengths are directed.

13. (Mixtilinear Incircles) Let $ABC$ be a triangle with circumcircle $\Gamma$ and let $\omega$ be a circle tangent internally to $\Gamma$ and to $AB$ anc $AC$ at $X$ and $Y$. Then the incenter of $ABC$ is the midpoint of segment $XY$.

14. (Curvilinear Incircles) Let $ABC$ be a triangle with circumcircle $\Gamma$ and let $D$ be a point on segment $BC$. Let $\omega$ be a circle tangent to $\Gamma$, $DA$ and $DC$. If $\omega$ is tangent to $DA$ and $DC$ at $F$ and $E$, then the incenter of $ABC$ lies on $FE$.